

8.2 – Compositions and Inverse Transformations

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W .

OR $T : V \rightarrow W$ is one-to-one if and only if for each vector $\mathbf{w}$ in the range of T , there is exactly one vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.

OR $T : V \rightarrow W$ is one-to-one if and only if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be **onto** (or **onto W**) if every vector in W is the image of at least one vector in V .

Examples:

One-to-one and onto

- Rotation operators on $\mathbb{R}^2$ ($0 \leq \theta < 2\pi$)
- $T : P_3 \rightarrow \mathbb{R}^4$ where $T(a+bx+cx^2+dx^3) = (a, b, c, d)$
- $T : M_{22} \rightarrow \mathbb{R}^4$ where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$

One-to-one but not onto

- $T : P_n \rightarrow P_{n+1}$, where $T(\vec{p}) = T(p(x)) = xp(x)$

Not one-to-one

- $D : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$, where

$$D(\vec{f}) = f'(x)$$

$$\text{Ex: } D(x^2) = D(x^2+1) = 2x$$

Theorem 8.2.1 If $T : V \rightarrow W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

pf: ($\Rightarrow$) By Thm 8.1.1, $T(\vec{0}) = \vec{0}$ because T is linear. If T is one-to-one, then if $T(\vec{v}) = \vec{0}$, then $\vec{v} = \vec{0}$.

($\Leftarrow$) Let $\vec{u}, \vec{v} \in V$ such that $\vec{u} \neq \vec{v}$.

$$\text{Then } \vec{u} - \vec{v} \neq \vec{0} \Rightarrow T(\vec{u}) - T(\vec{v}) \neq \vec{0}$$

$$\Rightarrow T(\vec{u}) \neq T(\vec{v}) \Rightarrow T \text{ is one-to-one.}$$

#4 Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, where $T(x, y) = (x - y, y - x, 2x - 2y)$

b. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $T(x, y) = (0, 2x + 3y)$

c. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $T(x, y) = (x + y, x - y)$

a) $(x, y) \in \ker(T)$ if $x - y = 0, y - x = 0, 2x - 2y = 0$
 $\Rightarrow x = y$. Not one-to-one.

b) $\ker(T) = \{(x, y) \mid 2x + 3y = 0\}$
 $y = -\frac{2}{3}x \rightarrow$ not one-to-one

c) $x + y = 0 \Rightarrow y = 0$
 $x - y = 0 \quad \uparrow$
 $2x = 0 \Rightarrow x = 0$

This is one-to-one.

Theorem 8.2.X If $T : V \rightarrow W$ is a one-to-one linear transformation and $\{v_1, v_2, \dots, v_k\}$ is a linearly independent subset of V , then $\{T(v_1), T(v_2), \dots, T(v_k)\}$ is linearly independent.

pf: Suppose $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_k T(\vec{v}_k) = \vec{0}$.

Because T is linear, this gives

$$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k) = \vec{0}$$

Since T is one-to-one, $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = \vec{0}$.

Then $c_i = 0$, $1 \leq i \leq k$ because

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent.

Therefore $\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)\}$ is linearly independent.

#5 Determine whether multiplication by A is one-to-one by computing the nullity of A and then applying Theorem 8.2.1

$$\text{a. } A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$$

$$\text{b. } A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$$

$$\text{a) } \text{rank}(A) = 1$$

$$n = 4, m = 3$$

$$\Rightarrow \text{nullity}(A) = 1$$

$$\Rightarrow \text{rank}(A) \leq 3$$

$$(\text{rank} + \text{nullity} = \# \text{ columns})$$

$$\Rightarrow \text{nullity}(A) \leq 1$$

not one-to-one.

not one-to-one.

Theorem 8.2.2 If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent.

a) T is one-to-one.

b) $\ker(T) = \{0\}$.

c) T is onto [i.e., $R(T) = W$].

Theorem 8.2.3 If $T_A : R^n \rightarrow R^m$ is a matrix transformation, then

a) T_A is one-to-one if and only if the columns of A are linearly independent.

b) T_A is onto if and only if the columns of A span R^m .

#14 Use Theorem 8.2.3 to determine whether multiplication by A is one-to-one, onto, both, or neither. Justify your answer.

$$\text{a. } A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\text{b. } A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$$

$$\text{c. } A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$$

$$\text{d. } A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 8.2.4 Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)

If A is an $n \times n$ matrix, then the following statements are equivalent.

- a) A is invertible.
- b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- c) The reduced row echelon form of A is I_n .
- d) A is expressible as a product of elementary matrices.
- e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
- f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
- g) $\det(A) \neq 0$.
- h) The column vectors of A are distinct and linearly independent.
- i) The row vectors of A are distinct and linearly independent.

- j) The column vectors of A span R^n .
- k) The row vectors of A span R^n .
- l) The column vectors of A form a basis for R^n .
- m) The row vectors of A form a basis for R^n .
- n) A has rank n .
- o) A has nullity 0.
- p) The orthogonal complement of the null space of A is R^n .
- q) The orthogonal complement of the row space of A is $\{0\}$.
- r) $\lambda = 0$ is not an eigenvalue of A . [This is from Theorem 5.1.5.]
- s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]
- t) The kernel of T_A is $\{0\}$.
- u) The range of T_A is R^n .
- v) T_A is one-to-one.

Definition: If $T : V \rightarrow W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then there is exactly one vector $\mathbf{v}$ in V for which $T(\mathbf{v}) = \mathbf{w}$. The **inverse of T** maps $\mathbf{w}$ back into $\mathbf{v}$ and is denoted by T^{-1} . (This is analogous to **inverse** in Section 1.9.)

Definition: (analogous to **composition** in Section 1.9)

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then the **composition** of T_2 with T_1 , denoted by $T_2 \circ T_1$ (which is read “ T_2 circle T_1 ”) is the function defined by the formula $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in U .

Theorem 8.2.5 If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then $(T_2 \circ T_1) : U \rightarrow W$ is also a linear transformation.

Theorem 8.2.6 If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are one-to-one linear transformations, then:

- a) $T_2 \circ T_1$ is one-to-one.
- b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

#38 Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for

a. $\mathbf{f}(x) = x^2 + 3x + 2$

b. $\mathbf{f}(x) = \sin x$

Key concepts from 8.3:

Definition 1: A linear transformation $T : V \rightarrow W$ that is both one-to-one and onto is said to be an **isomorphism**, and W is said to be **isomorphic** to V .

Theorem 8.3.1 Every real n -dimensional vector space is isomorphic to R^n . [In this sense, every n -dimensional vector space is algebraically equivalent to R^n .]

Theorem 8.3.2 If S is an ordered basis for a vector space V , then the coordinate map $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$ is an isomorphism between V and R^n .

Theorem 8.3.X A linear transformation $T : V \rightarrow W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for V , $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is a basis for W .

Examples of Natural isomorphisms

Between P_{n-1} and R^n : $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \dots, a_{n-1})$

Between M_{22} and R^4 : $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a, b, c, d)$